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DOUBLE STAGE SHRINKAGE TESTIMATION IN EXPONENTIAL TYPE-II CENSORED DATA

Gyan Prakash[1], **D. C. Singh**[2]

ABSTRACT

The present paper investigates the properties of the shrinkage testimators for mean and variance of an Exponential distribution in double stage samples, by using the cost function when Type – II censored data are available.

Key words: Type-II censored data; Shrinkage factor; Shrinkage testimator; Level of significance; Effective Interval; Cost function.

1. Introduction

The most widely used lifetime distribution is the Exponential distribution, having the probability density function for any random variable x

$$f(x;\lambda)=\frac{1}{\lambda}e^{-\frac{x}{\lambda}} \quad ; \; x\geq 0,\; \lambda\geq 0. \tag{1.1}$$

Here, the parameter λ is called as the scale parameter, better known as the average life and λ^2 is called as the variance. The mean time to failure is λ and $\bar{x}$ is an unbiased estimate for the parameter λ. Further, the survival function is $S(x)=e^{-\lambda x}$ and failure rate is $\gamma = 1/\lambda$. When the failure rate is constant, the Exponential model has been found to be useful.

In life testing, fatigue failures and other kinds of destructive test situations, the observations usually occurred in ordered manner in such a way that weakest items failed first and then the second one and so on. Let $x_{1i}\,(i=1,2,...,r_1)$ and $x_{2j}\,(j=1,2,...,r_2)$ be the two independent random samples of size r_1 and r_2

[1] Department of Statistics, Harish Chandra P. G. College, Varanasi, U. P., India. E-Mail: ggyanji@yahoo.com
[2] Department of Statistics, Harish Chandra P. G. College, Varanasi, U. P., India.

respectively, drawn from an Exponential population with mean λ and variance λ^2. Epstien & Sobel (1953) proved that the total test time

$$T_1 = \sum_{i=1}^{r_1} x_{(1i)} + (n - r_1)\, x_{(1r_1)} \;; n > r_1 \tag{1.2}$$

is complete sufficient statistic for λ and r_1 is the uncensored sample size. The minimum variance unbiased (MVU) estimator of λ is $\frac{T_1}{r_1}$ and for the variance λ^2 is $\frac{T_1^2}{r_1(r_1+1)}$. Further, the minimum mean square error (MMSE) estimator for the mean and the variance are given respectively as

$$\frac{T_1}{r_1+1}$$

and

$$\frac{T_1^2}{(r_1+2)(r_1+3)}.$$

The unbiased pooled estimates for λ and λ^2 are respectively given as

$$T_M = \frac{T_1 + T_2}{r_1 + r_2}$$

and

$$T_V = \frac{T_1^2 + T_2^2}{r_1 (r_1+1) + r_2 (r_2+1)}.$$

It has been obtained that the estimator, by using the preliminary testimator, adaptive testimator, conditional testimator or shrinkage testimator, is superior to the existing estimator if the sample size and level of significance are small. Bancroft & Han (1977) and Han et al. (1988) described the inference based on conditional specification. Pandey (1983), Ebrahimi & Hosmane (1987), Pandey & Srivastava (1987) and Pandey (1997) described shrinkage estimation under Exponential data.

Many authors including Stein (1945), Weiss (1955) and Chapman (1960) have used two stage samples. These authors used the first sample to make inference on the unknown nuisance parameter and then used it in conjunction with the second to estimate the parameter of interest. Katti (1962) and Shah (1964) used a two stage estimate, but used the first stage to test the accuracy of the prior information (guess estimate) and if it is accurate, used first stage sample to estimate the parameter, otherwise used it in conjunction with the second sample to estimate the parameter. Al–Bayyati & Arnold (1969, 70, 72) proposed the double stage shrinkage estimators for the parameter using the first stage sample to test the accuracy of the guess value and if it is accurate, they used shrinkage of the usual estimator towards guess value, otherwise used the first sample in conjunction with a new second sample to estimate the parameter. Further, Pandey (1979) introduced the preliminary test in constructing the acceptance region in double stage shrinkage estimation. Waiker et al. (1984) and Adke et al. (1987) proposed two stage shrinkage testimators depending upon the outcome of the preliminary test of hypothesis.

The proposed double stage technique is to obtain first a sample of size r_1 and compute the usual estimate for the parameter. If the usual estimate implies that our prior estimate or guess value of the parameter was reasonable, we stop the sampling and estimate the parameter by using the shrinkage estimator. Otherwise, we obtain r_2 additional observations and then use an improved estimate based on all $(r_1 + r_2)$ observations.

In the present paper, we study the performance of the shrinkage testimators in double stage samples under the invariant form of the LINEX loss function for the mean and variance of the Exponential distribution when Type – II censored data are available.

2. The proposed class of estimators

The proposed class of estimators for the parameters λ and λ^2 based on two samples of sizes r_1 and r_2 are given as respectively

$$P_M = l_1 T_M \,;\; l_1 \in R^+ \tag{2.1}$$

and

$$P_V = l_2 T_V \,;\; l_2 \in R^+ . \tag{2.2}$$

The invariant form of the LINEX loss function for any parameter μ (Basu & Ebrahimi, 1992) is given as

$$L(\Delta) = e^{a\Delta} - a\Delta - 1 \,;\, a \neq 0,\; \Delta = \left(\frac{\hat{\mu}}{\mu} - 1\right), \tag{2.3}$$

where $'a'$ is the shape parameter and $\hat{\mu}$ is an estimate of the parameter μ.

The LINEX loss function is convex and its shape is to be determined by the value of $'a'$ (the sign of $'a'$ reflects the direction of asymmetry, $a > 0\,(a < 0)$ if over estimation is more (less) serious than under estimation) and its magnitude reflects the degree of asymmetry. The LINEX loss function has been found to be appropriate in the situations where overestimation is more serious than underestimation and vice versa. The LINEX loss criterion will change to mean square error criterion if $|a| \to 0$.

The value of constant l_1 for which the risk of P_M is minimum under the LINEX loss function (2.3) is

$$\hat{l}_1 = \frac{r_1 + r_2}{a}\left(1 - e^{-a/r_1+r_2+1}\right).$$

Similarly, the value of $l_2 = \hat{l}_2$ (say) for which the risk of P_V is minimum under the loss (2.3), is obtained by solving the given equality

$$e^{a}\left(\Gamma(r_1+1)\Gamma r_2 + \Gamma r_1\,\Gamma(r_2+1)\right) = I\left(0, \infty, 0, \infty, e^{a\Delta_V}\right),$$

$$\text{where } \Delta_V = \frac{l_2\left(x^2 + y^2\right)}{r_1(r_1+1) + r_2(r_2+1)},$$

$$I\left(x_1, x_2, y_1, y_2, w\right) = \int_{x_1}^{x_2}\int_{y_1}^{y_2} \frac{(w)\,e^{-x}\,e^{-y}}{\Gamma r_1\,\Gamma r_2}$$

$x^{r_1-1}\,y^{r_2-1}\,dx\,dy$ and w may be the function of x and y.

Hence, the improved class of estimators for the parameters λ and λ^2 are

$$\hat{P}_M = \hat{l}_1\,T_M$$

and

$$\hat{P}_V = \hat{l}_2\,T_V$$

with the risk

$$R\left(\hat{P}_M\right) = \left(1 + r_1 + r_2\right)\left(e^{-a/\,1+r_1+r_2} - 1\right) + a$$

and

$$R\left(\hat{P}_V\right) = I\left(0, \infty, 0, \infty, e^{a\Delta_{V0}}\right) - a\,I\left(0, \infty, 0, \infty, \Delta_{V0}\right) - 1,$$

where $\Delta_{V0} = \dfrac{\hat{l}_2\left(x^2 + y^2\right)}{r_1(r_1+1) + r_2(r_2+1)}$.

3. The proposed shrinkage estimators

The shrinkage estimator (Thompson, 1968) for the mean λ and variance λ^2 based on the sample of size r_1, when prior point estimate of the parameters is available, are proposed:

$$T_{SM} = \lambda_0 + k\left(T_1 - \lambda_0 r_1\right) r_1^{-1} \tag{3.1}$$

and

$$T_{SV} = \lambda_0^2 + k\left(\frac{T_1^2}{r_1(r_1+1)} - \lambda_0^2\right);\ 0 \le k \le 1. \tag{3.2}$$

A value of shrinkage factor k near to zero implies strong belief in the guess value (prior point value) and near to one implies strong belief in the sample values. The risks of these shrinkage estimator under the loss (2.3) are obtained as

$$R\left(T_{SM}\right) = e^{a(\delta-1)} e^{-ak\delta}\left(1 - a k r_1^{-1}\right)^{-r_1} - a\left(\delta-1\right)\left(1-k\right) - 1$$

and

$$R\left(T_{SV}\right) = e^{a\left(\delta^2-1\right)} G\left(0, \infty, e^{a\Delta_{V1}}\right) - a\left(\delta^2-1\right)\left(1-k\right) - 1,$$

where $\Delta_{V1} = k\left(\dfrac{x^2}{r_1\left(r_1+1\right)} - \delta^2\right)$, $G\left(x_1, x_2, u\right) = \displaystyle\int_{x_1}^{x_2} \frac{u}{\Gamma r_1} e^{-x} x^{r_1-1}\, dx$,

$\delta = \dfrac{\lambda_0}{\lambda}$ and u may be the function of x.

4. Shrinkage testimators for the mean and their properties

Several researchers have studied the performance of the shrinkage estimators and found that the shrinkage estimator performs better with respect to any usual estimator or improved estimator when the guess value of the parameter is approximately equal to the true value of the parameter and the sample size is small. This implies that we may test the hypothesis $H_0: \delta = 1$ against $H_1: \delta \ne 1$. A test statistic $\left(2T_1/\lambda\right) \sim \chi^2_{(2r_1)}$ is available for testing the hypothesis H_0 against H_1. The proposed double stage shrinkage testimator for the mean λ is defined as

$$\hat{\lambda}_1 = \begin{cases} T_{SM} & \text{if } \left(t_1 \le T_1 \le t_2 \right) \\ \hat{P}_M & \text{else} \end{cases},$$

where $t_i = \dfrac{m_i \lambda_0}{2}; i = 1, 2$. Here m_1 and m_2 are the values of the lower and upper $100\dfrac{\alpha}{2}\%$ points of the chi – square distribution with $2r_1$ degrees of freedom.

The relative bias of the shrinkage testimator $\hat{\lambda}_1$ is obtained as

$$RB\left(\hat{\lambda}_1\right) = E\left(\hat{\lambda}_1\right)\lambda^{-1} - 1$$

$$= G\left(x_1, x_2, \left(\Delta_{M1} + \delta\right)\right) - I\left(x_1, x_2, 0, \infty, \Delta_{M0}\right) + \hat{l}_1 - 1, \qquad (4.1)$$

where $\Delta_{M1} = k\left(x - \delta r_1\right) r_1^{-1}$, $\Delta_{M0} = \hat{l}_1 \left(r_1 + r_2\right)^{-1}\left(x + y\right)$.

The risk under the LINEX loss (2.3) for $\hat{\lambda}_1$ is

$$R\left(\hat{\lambda}_1\right) = e^{-a(\delta-1)} G\left(x_1, x_2, e^{a\Delta_{M1}}\right) - a\,G\left(x_1, x_2, \left(\Delta_{M1} + \delta\right)\right)$$
$$- e^{-a}\, I\left(x_1, x_2, 0, \infty, e^{a\Delta_{M0}}\right) + a\, I\left(x_1, x_2, 0, \infty, \Delta_{M0}\right)$$
$$+ \left(1 + r_1 + r_2\right)\left(e^{-a/1+r_1+r_2} - 1\right) + a. \qquad (4.2)$$

Our problem is considered as a sequential estimation problem with stopping random variable R defined as

$$R = \begin{cases} r_1 & \text{if } t_1 \le T_1 \le t_2 \\ r_1 + r_2 & \text{otherwise} \end{cases}. \qquad (4.3)$$

Let us introduce a cost $c\,(> 0)$ also, for each observation. Then the risk of $\hat{\lambda}_1$ and $\hat{P}_M$ are

$$\tilde{R}\left(\hat{\lambda}_1\right) = R\left(\hat{\lambda}_1\right) + c\,E\left(R\right)$$

and

$$\tilde{R}\left(\hat{P}_M\right) = R\left(\hat{P}_M\right) + c\left(r_1 + r_2\right).$$

Therefore, the relative efficiency of $\hat{\lambda}_1$ with respect to $\hat{P}_M$ is given by

$$RE\left(\hat{\lambda}_1, \hat{P}_M\right) = \tilde{R}\left(\hat{P}_M\right) / \tilde{R}\left(\hat{\lambda}_1\right).$$

The expressions of the relative bias and relative efficiency are the functions of r_1, r_2, δ, a, c, k and α (level of significance). For the selected values of $r_1, r_2 = 03, 05, 07$; $\delta = 0.25\,(0.25)\,1.75$; $a = 0.25, 0.50, 1.00$; $k = 0.25, 0.50, 0.75$; $c = 0.50, 05, 10, 50$ and $\alpha = 0.01, 0.05, 0.20$, the relative biases (not presented here), and the relative efficiency have been calculated. However, the relative efficiencies are presented only for $r_1 = 03, c = 0.50, \alpha = 0.01$ in Table 01.

We observe that the relative biases are small and lie between -0.140 and 0.281. The biases are negative for $\delta < 1.00$ and positive otherwise. The value of biases decreases as a, k or α increases when $\delta \geq 1.00$. The biases decrease as r_2 increases in the interval $0.50 \leq \delta \leq 1.25$ and increases as r_1 increases for all considered values of δ.

The testimator $\hat{\lambda}_1$ performs well with respect to $\hat{P}_M$ in the effective interval $0.50 \leq \delta \leq 1.25$ and the effective interval decreases with increase in r_1 and attains maximum efficiency at the point $\delta = 1.00$. The efficiency increases as r_2 increases for all considered values of δ but it decreases as r_1 or 'a' increases except for $\delta = 1.00$. The efficiency also increases as per unit cost c increases in the interval $0.75 \leq \delta \leq 1.25$ for small k, r_1 and α, and decreases otherwise. However, the gain in efficiency is nominal. In addition, as level of significance α increases, the efficiency decreases.

It may interest one to obtain the value of the shrinkage factor k that minimizes the risk of $\hat{\lambda}_1$ (4.2). For this the minimum value of $k = \hat{k}_M$ (say) is obtained numerically by solving the given equality

$$e^{-a(\delta-1)}\, G\left(x_1, x_2, \left(\Delta_{M1}\, e^{a\Delta_{M1}}\right)\right) = G\left(x_1, x_2, \Delta_{M1}\right). \tag{4.4}$$

Hence, the proposed double stage shrinkage testimator for the mean is

$$\hat{\lambda}_2 = \begin{cases} \lambda_0 + \hat{k}_M\left(T_1 - \lambda_0 r_1\right) r_1^{-1} & \text{if } \left(t_1 \leq T_1 \leq t_2\right) \\ \hat{P}_M & \text{else} \end{cases}.$$

Thus, the expression of the relative bias and risk under the LINEX loss are obtained as

Table 1. RE $\left(\hat{\lambda}_1,\ \hat{P}_M\right)$

$\alpha = 0.01, c = 0.50$									
$k = 0.25$			δ						
r_1	r_2	a	0.25	0.50	0.75	1.00	1.25	1.50	1.75
03	03	0.25	1.1204	1.9647	2.3878	4.5153	2.1638	2.1749	2.1641
		0.50	1.1154	1.9525	2.3786	4.5382	2.1561	2.1467	2.0846
		1.00	1.0995	1.9134	2.3639	4.6256	2.1493	2.0262	1.7861
	05	0.25	1.1558	2.2141	2.9266	5.8443	2.8566	2.8894	2.8800
		0.50	1.1510	2.1987	2.9209	5.8574	2.8509	2.8424	2.7647
		1.00	1.1350	2.1469	2.9019	5.9074	2.8226	2.6501	2.3386
	07	0.25	1.1783	2.3974	3.3983	7.1011	3.5378	3.6008	3.5955
		0.50	1.1739	2.3811	3.3887	7.1076	3.5255	3.5366	3.4458
		1.00	1.1592	2.3255	3.3549	7.1320	3.4711	3.2775	2.8963
$k = 0.50$									
03	03	0.25	1.1206	1.9654	2.3766	4.5081	2.1609	2.1767	2.1747
		0.50	1.1161	1.9553	2.3740	4.5097	2.1540	2.1527	2.1255
		1.00	1.1018	1.9231	2.3679	4.5117	2.1164	2.0407	1.9149
	05	0.25	1.1560	2.2148	2.9252	5.8353	2.8528	2.8917	2.8942
		0.50	1.1516	2.2014	2.9157	5.8216	2.8350	2.8504	2.8190
		1.00	1.1369	2.1563	2.8838	5.7654	2.7541	2.6690	2.5073
	07	0.25	1.1785	2.3980	3.3968	7.0905	3.5331	3.6036	3.6132
		0.50	1.1744	2.3836	3.3831	7.0652	3.5060	3.5466	3.5134
		1.00	1.1608	2.3343	3.3354	6.9649	3.3875	3.3009	3.1051
$k = 0.75$									
03	03	0.25	1.1208	1.9658	2.3741	4.4963	2.1541	2.1719	2.1749
		0.50	1.1168	1.9568	2.3644	4.4626	2.1264	2.1326	2.1249
		1.00	1.1040	1.9290	2.3338	4.3301	2.0063	1.9562	1.8990
	05	0.25	1.1561	2.2152	2.9224	5.8204	2.8439	2.8853	2.8944
		0.50	1.1521	2.2028	2.9048	5.7624	2.7989	2.8238	2.8181
		1.00	1.1387	2.1619	2.8452	5.5388	2.6117	2.5587	2.4864
	07	0.25	1.1786	2.3983	3.3937	7.0728	3.5222	3.5957	3.6134
		0.50	1.1749	2.3850	3.3713	6.9954	3.4618	3.5135	3.5123
		1.00	1.1623	2.3396	3.2936	6.6976	3.2135	3.1647	3.0793

$$RB\left(\hat{\lambda}_2\right) = G\left(x_1, x_2, \left(\Delta_{M2} + \delta\right)\right) - I\left(x_1, x_2, 0, \infty, \Delta_{M0}\right) + \hat{l}_1 - 1$$

and

$$R\left(\hat{\lambda}_2\right) = e^{-a(\delta-1)} G\left(x_1, x_2, e^{a\Delta_{M2}}\right) - a\,G\left(x_1, x_2, \left(\Delta_{M2} + \delta\right)\right)$$
$$- e^{-a}\, I\left(x_1, x_2, 0, \infty, e^{a\Delta_{M0}}\right) + a\, I\left(x_1, x_2, 0, \infty, \Delta_{M0}\right)$$
$$+ \left(1 + r_1 + r_2\right)\left(e^{-a/1+r_1+r_2} - 1\right) + a \;;\; \Delta_{M2} = \hat{k}_M \left(x - \delta r_1\right) r_1^{-1}.$$

The relative efficiency of $\hat{\lambda}_2$ with respect to $\hat{P}_M$ is defined as

$$RE\left(\hat{\lambda}_2, \hat{P}_M\right) = \tilde{R}\left(\hat{P}_M\right) / \tilde{R}\left(\hat{\lambda}_2\right).$$

The expressions of relative bias and the relative efficiency are the functions of the r_1, r_2, δ, a, c and α. For the similar set of selected values, the relative bias (not presented here) and the relative efficiency (presented in Table 02 for $r_1 = 03, c = 0.50, \alpha = 0.01$ only) have been calculated.

The relative biases are small and lie between -0.141 and 0.320. The bias increases as r_1 increases for $\delta \le 0.75$. The shrinkage testimator $\hat{\lambda}_2$ is more efficient than $\hat{P}_M$ in the effective interval $0.50 \le \delta \le 1.50$. Other properties of the shrinkage testimator $\hat{\lambda}_2$ related to the bias and the relative efficiency are similar to $\hat{\lambda}_1$. Based on the gain in relative efficiency, the shrinkage testimator $\hat{\lambda}_2$ is preferred over $\hat{\lambda}_1$ in the interval $0.50 \le \delta \le 1.50$.

5. Shrinkage testimators for the variance and their properties

The proposed double stage shrinkage testimator for the variance λ^2 is defined as

$$\hat{\lambda}_3 = \begin{cases} T_{SV} & \text{if } \left(t_1 \le T_1 \le t_2\right) \\ \hat{P}_V & \text{else} \end{cases}.$$

The expressions of the relative bias and risk under the LINEX loss for the proposed double stage shrinkage testimator $\hat{\lambda}_3$ are obtained as

$$RB\left(\hat{\lambda}_3\right) = G\left(x_1, x_2, \left(\Delta_{V1} + \delta^2\right)\right) - I\left(x_1, x_2, 0, \infty, \Delta_{V0}\right) + \hat{l}_2 - 1$$

and

$$R\left(\hat{\lambda}_3\right) = e^{-a\left(\delta^2-1\right)} G\left(x_1, x_2, e^{a\Delta_{V1}}\right) - a\,G\left(x_1, x_2, \left(\Delta_{V1} + \delta^2\right)\right)$$
$$-e^{-a}\, I\left(x_1, x_2, 0, \infty, e^{a\Delta_{V0}}\right) + a\,I\left(x_1, x_2, 0, \infty, \Delta_{V0}\right)$$
$$+\ e^{-a}\, I\left(0, \infty, 0, \infty, e^{a\Delta_{V0}}\right) + a\left(1-\hat{l}_2\right) - 1. \qquad (5.1)$$

Now, the risks under the sequential estimation problem with per unit cost $c\,(>0)$ are given as

$$\widetilde{R}\left(\hat{\lambda}_3\right) = R\left(\hat{\lambda}_3\right) + c\,E\left(R\right)$$

Table 2. $RE\left(\hat{\lambda}_2,\ \hat{P}_M\right)$

$\alpha = 0.01\ c = 0.50$									
			δ						
r_1	r_2	a	0.25	0.50	0.75	1.00	1.25	1.50	1.75
03	03	0.25	1.1205	2.1883	2.6471	5.0275	2.4012	2.4002	2.1368
		0.50	1.1158	2.1741	2.6433	5.0465	2.3737	2.2894	1.9497
		1.00	1.1010	2.1287	2.6328	5.1163	2.2367	1.7896	1.1065
	05	0.25	1.1559	2.4662	3.2582	6.5074	3.1702	3.1887	2.8438
		0.50	1.1514	2.4484	3.2466	6.5139	3.1245	3.0317	2.5860
		1.00	1.1363	2.3889	3.2068	6.5351	2.9116	2.3412	1.4489
	07	0.25	1.1784	2.6702	3.7834	7.9068	3.9263	3.9738	3.5503
		0.50	1.1743	2.6516	3.7671	7.9045	3.8645	3.7726	3.2231
		1.00	1.1603	2.5879	3.7094	7.8913	3.5825	2.8966	1.7946
05	03	0.25	1.0207	1.7685	2.0997	4.0534	1.9123	1.8461	1.5947
		0.50	1.0135	1.7311	2.0923	4.1851	1.8601	1.5850	1.2338
		1.00	0.9957	1.6429	2.0857	4.6473	1.6676	1.3321	0.6149
	05	0.25	1.0286	1.8787	2.4058	4.8907	2.3563	2.2881	1.9790
		0.50	1.0207	1.8291	2.3705	4.9772	2.2553	1.9293	1.5023
		1.00	0.9989	1.6990	2.2756	5.2884	1.9201	1.3819	0.7053
	07	0.25	1.0342	1.9621	2.6690	5.6836	2.7951	2.7303	2.3647
		0.50	1.0264	1.9086	2.6164	5.7377	2.6512	2.2787	1.7757
		1.00	1.0038	1.7592	2.4635	5.9364	2.1874	1.4326	0.8035
07	03	0.25	0.9924	1.5275	1.8722	4.2518	1.6202	1.2577	0.7578
		0.50	0.9847	1.4370	1.8619	5.4694	1.4852	1.2272	0.4461
		1.00	0.9708	1.4037	1.8618	7.0728	1.3231	1.1383	0.2929
	05	0.25	0.9937	1.5376	1.9383	4.6493	1.7910	1.2734	0.8327
		0.50	0.9886	1.4961	1.8790	5.6253	1.4925	1.2422	0.4390
		1.00	0.9793	1.4093	1.8715	7.2772	1.3478	1.2267	0.2632
	07	0.25	0.9962	1.5576	2.0320	5.0534	1.9745	1.3006	0.9164
		0.50	0.9953	1.5392	1.8863	5.8119	1.5244	1.2954	0.4426
		1.00	0.9804	1.5370	1.8717	7.6954	1.3776	1.2417	0.2433

and

$$\widetilde{R}\left(\hat{P}_V\right) = R\left(\hat{P}_V\right) + c\left(r_1 + r_2\right)$$

The relative efficiencies of $\hat{\lambda}_3$ with respect to $\hat{P}_V$ is defined as

$$RE\left(\hat{\lambda}_3, \hat{P}_V\right) = \widetilde{R}\left(\hat{P}_V\right) / \widetilde{R}\left(\hat{\lambda}_3\right).$$

For the similar set of selected values as considered in previous section, the relative bias (not presented here) and the relative efficiency have been calculated. The relative efficiencies are presented only for $r_1 = 03, c = 0.50, \alpha = 0.01$ in the Table 03.

Table 3. Table 03 :: RE $\left(\hat{\lambda}_3, \hat{P}_V\right)$

$\alpha = 0.01, c = 0.50$									
$k = 0.25$			δ						
r_1	r_2	a	0.25	0.50	0.75	1.00	1.25	1.50	1.75
03	03	0.25	1.1210	1.9652	2.4789	4.5201	2.1606	2.1465	2.0756
		0.50	1.1178	1.9552	2.3829	4.5584	2.1516	2.0258	1.7409
		1.00	1.1109	1.9527	2.3787	4.9161	2.1198	1.6419	1.1344
	05	0.25	1.1563	2.2142	2.9269	5.8497	2.8522	2.8512	2.7616
		0.50	1.1529	2.1996	2.9230	5.8792	2.8307	2.6810	2.3069
		1.00	1.1423	2.1549	2.9148	5.9953	2.7237	2.0469	1.1672
	07	0.25	1.1787	2.3971	3.3979	7.1062	3.5317	3.5526	3.4470
		0.50	1.1755	2.3807	3.3880	7.1276	3.4983	3.3338	2.8731
		1.00	1.1652	2.3285	3.3566	7.2092	3.3401	2.5249	1.4411
$k = 0.50$									
03	03	0.25	1.1211	1.9657	2.4649	4.5103	2.1568	2.1577	2.1257
		0.50	1.1180	1.9569	2.3787	4.5181	2.1320	2.0599	1.9022
		1.00	1.1114	1.9384	2.3775	4.7376	2.0734	1.6366	1.0795
	05	0.25	1.1564	2.2146	2.9256	5.8374	2.8471	2.8660	2.8283
		0.50	1.1531	2.2013	2.9182	5.8287	2.8051	2.7260	2.5206
		1.00	1.1428	2.1604	2.8993	5.7827	2.5706	2.0403	1.3483
	07	0.25	1.1788	2.3975	3.3965	7.0916	3.5255	3.5711	3.5302
		0.50	1.1756	2.3823	3.3828	7.0680	3.4669	3.3897	3.1393
		1.00	1.1656	2.3337	3.3400	6.9600	3.1535	2.5168	1.6648
$k = 0.75$									
03	03	0.25	1.1211	1.9660	2.4356	4.4938	2.1448	2.1504	2.1384
		0.50	1.1182	1.9582	2.3753	4.4500	2.0763	2.0096	1.9066
		1.00	1.1120	1.9228	2.3704	4.4347	1.7768	1.2574	1.1751
	05	0.25	1.1564	2.2150	2.9231	5.8166	2.8315	2.8564	2.8451
		0.50	1.1532	2.2025	2.9087	5.7431	2.7324	2.6596	2.5265
		1.00	1.1433	2.1647	2.8668	5.4211	2.2054	1.5685	1.2809
	07	0.25	1.1788	2.3978	3.3937	7.0670	3.5063	3.5591	3.5512
		0.50	1.1757	2.3834	3.3726	6.9669	3.3777	3.3072	3.1466
		1.00	1.1660	2.3378	3.3051	6.5351	2.7080	1.9352	1.3876

We observed that the relative biases are negligibly small and lie between -0.104 and 0.170. The biases are positive for $\delta \geq 1.00$ and negative otherwise. The values of bias decrease (increase) as 'a' (r_1) increases in the interval $\delta \leq 1.50$. The biases also decrease as level of significance increases in the interval $\delta \geq 0.75$. The bias decreases when the shrinkage factor k increases in the range $\delta \geq 1.00$ for small $\alpha (= 0.01)$ and in the range $0.25 \leq \delta \leq 1.75$ for large α.

The shrinkage testimator $\hat{\lambda}_3$ is more efficient than $\hat{P}_V$ in the interval $0.25 \le \delta \le 1.50$ and the effective interval decreases with the shrinkage factor k or per unit cost c increase. The efficiency attains maximum at the point $\delta = 1.00$. The efficiency increases (decreases) as r_2 (r_1) increases for all considered values of δ. The decreasing trend in the efficiency is also seen when α increases when $\delta \le 1.25$ and when 'a' increase for all δ (except for $\delta = 1.00$). Around $\delta = 1.00,$ the efficiency also decreases with the shrinkage factor k increase. The nominal gain in efficiency is recorded when per unit cost c increases for all considered values of parametric space when $0.75 \le \delta \le 1.25$.

The value of $k = \hat{k}_V$ (say) that minimizes the risk of $\hat{\lambda}_3$ (5.1) can be obtained numerically by solving the given equality

$$e^{-a\left(\delta^2-1\right)} G\left(x_1, x_2, \left(\Delta_{V1} e^{a\Delta_{V1}}\right)\right) = G\left(x_1, x_2, \Delta_{V1}\right). \tag{5.2}$$

Hence, the improved double stage shrinkage testimator of the variance is

$$\hat{\lambda}_4 = \begin{cases} \lambda_0^2 + \hat{k}_V \dfrac{\left(T_1^2 - \lambda_0^2 r_1(r_1+1)\right)}{r_1(r_1+1)} & \text{if } \left(t_1 \le T_1 \le t_2\right) \\ \hat{P}_V & \text{else} \end{cases}$$

with relative bias

$$RB\left(\hat{\lambda}_4\right) = G\left(x_1, x_2, \left(\Delta_{V2} + \delta^2\right)\right) - I\left(x_1, x_2, 0, \infty, \Delta_{V0}\right) + \hat{l}_2 - 1$$

and the risk

$$\begin{aligned} R\left(\hat{\lambda}_4\right) = {} & e^{-a\left(\delta^2-1\right)} G\left(x_1, x_2, e^{a\Delta_{V2}}\right) - a\,G\left(x_1, x_2, \left(\Delta_{V2} + \delta^2\right)\right) \\ & - e^{-a}\, I\left(x_1, x_2, 0, \infty, e^{a\Delta_{V0}}\right) + a\, I\left(x_1, x_2, 0, \infty, \Delta_{V0}\right) \\ & + e^{-a}\, I\left(0, \infty, 0, \infty, e^{a\Delta_{V0}}\right) + a\left(1 - \hat{l}_2\right) - 1, \end{aligned} \tag{5.2}$$

where $\Delta_{V2} = \hat{k}_V \left(\dfrac{x^2}{r_1(r_1+1)} - \delta^2 \right).$

The relative efficiency of $\hat{\lambda}_4$ with respect to $\hat{P}_V$ is

$$RE\left(\hat{\lambda}_4, \hat{P}_V\right) = \tilde{R}\left(\hat{P}_V\right) / \tilde{R}\left(\hat{\lambda}_4\right).$$

For the similar set of selected values, the relative bias (not presented here) and the relative efficiency (presented in Table 04 only for $r_1 = 03, c = 0.50, \alpha$

$=0.01$) have been calculated. The relative biases are negligible small and lie between -0.105 and 0.178. The biases are positive for $\delta>1.00$ and negative for $\delta<1.00$. Other properties are similar to the double stage shrinkage testimator $\hat{\lambda}_3$.

The shrinkage testimator $\hat{\lambda}_4$ is more efficient than $\hat{P}_V$ for all considered values of the parametric space (Table 04). Other properties are similar to the double stage shrinkage testimator $\hat{\lambda}_3$ except when r_1 increases, the efficiency decreases when $\delta\le 1.25$.

Table 4. $RE\left(\hat{\lambda}_4,\ \hat{P}_V\right)$

$\alpha=0.01$ $c=0.50$									
			δ						
r_1	r_2	a	0.25	0.50	0.75	1.00	1.25	1.50	1.75
03	03	0.25	1.1210	1.9650	2.3783	5.5171	3.1376	2.5920	1.8252
		0.50	1.1179	1.9546	2.3775	5.5453	3.0334	2.4312	1.1778
		1.00	1.1112	1.9510	2.1597	5.8492	2.6652	1.0545	1.0000
	05	0.25	1.1564	2.2140	2.9253	6.8460	3.8220	2.7355	2.4286
		0.50	1.1530	2.1991	2.9169	6.8628	3.6762	2.4949	1.2356
		1.00	1.1426	2.1532	2.8935	6.9156	2.9809	1.0681	1.0000
	07	0.25	1.1788	2.3969	3.3961	8.1017	4.4947	3.4087	3.0316
		0.50	1.1756	2.3802	3.3814	8.1082	4.3088	2.5760	1.2936
		1.00	1.1654	2.3269	3.3337	8.1159	3.2380	1.0840	1.0000
05	03	0.25	1.0235	1.6007	1.9886	4.6037	2.7259	2.7178	1.6847
		0.50	1.0231	1.5986	1.9020	4.6222	2.7064	2.5983	1.3338
		1.00	1.0219	1.5932	1.9005	4.6997	2.5991	1.9100	1.0502
	05	0.25	1.0315	1.7027	2.1712	5.3684	3.1379	3.1413	2.1032
		0.50	1.0310	1.6990	2.1692	5.3823	3.1099	2.9883	1.6612
		1.00	1.0295	1.6876	2.1651	5.4393	2.9626	2.1225	1.0619
	07	0.25	1.0370	1.7784	2.4124	6.0896	3.5433	3.5634	2.5216
		0.50	1.0365	1.7741	2.4084	6.1011	3.5079	3.3779	1.9892
		1.00	1.0349	1.7603	2.3962	6.1476	3.1252	2.3370	1.0737
07	03	0.25	1.0046	1.4656	1.6869	4.2037	2.5430	2.5470	1.5368
		0.50	1.0045	1.4651	1.6785	4.2155	2.5360	2.4962	1.4250
		1.00	1.0043	1.4640	1.6263	4.2656	2.4998	2.2331	1.1683
	05	0.25	1.0063	1.5126	1.8455	4.7199	2.8346	2.8520	1.8425
		0.50	1.0063	1.5115	1.8451	4.7287	2.8240	2.7887	1.7058
		1.00	1.0060	1.5080	1.8444	4.7655	2.7720	2.4663	1.6096
	07	0.25	1.0076	1.5482	1.9886	5.2046	3.1214	2.9156	2.1482
		0.50	1.0075	1.5467	1.9871	5.2118	3.1080	2.8808	1.9870
		1.00	1.0072	1.5420	1.9828	5.2415	3.0431	2.4711	1.9398

Based on the magnitude of the relative efficiency, one may prefer the shrinkage testimator $\hat{\lambda}_4$ in the interval $1.00 \le \delta \le 1.50$ and $\hat{\lambda}_3$ otherwise.

REFERENCES

ADKE, S. R., WAIKAR, V. B. and SCHUURMANN, F. J. (1987). A two stage shrinkage testimator for the mean of an exponential distribution. Communication in Statistics – Theory and Methods, 16, 1821–1834.

AL-BAYYATI, H. A. and ARNOLD, J. C. (1969). Double stage shrunken estimator of the variance. Ind. Statist. Theory Meth. Assoc., 7, 175–184.

AL-BAYYATI, H. A. and ARNOLD, J. C. (1972). On double stage estimation in simple liner regression using prior knowledge. Technometrices, 14, 405–414.

AL-BAYYATI, H. A. and ARNOLD, J. C. (1970). On double stage estimation of mean using prior knowledge. Biometrics, 26, 787–800.

BANCROFT, T. A. and HAN, C. P. (1977). Inference based on conditional specification. A note and a bibliography. International Statistical Review, 15, 117–127.

BASU, A. P. and EBRAHIMI, N. (1991). Bayesian approach to life testing and reliability estimation using asymmetric loss function. Journal of Statistical Planning and Inferences, 29, 21–31.

CHAPMAN, D. G. (1960). Some two sample test. Annals of Mathematical Statistics, 21, 601–606.

EBRAHIMI, N. and HOSMANE, B. (1987). On shrinkage estimation of the Exponential parameter. Communications in Statistics –Theory and Methods, 16, 2623–2637.

EPSTEIN, B. and SOBEL, M. (1953). Life Testing. Journal of American Statistical Association, 48, 486–507.

HAN, C. P., RAO, C. V. and RAVICHANDRAN, J. (1988). Inference based on the condition specification: A second bibliography. Communications in Statistics –Theory and Methods, A–17, 1–21.

KATTI, S. K. (1962). Use of some apriori knowledge in the estimation of mean from double samples. Biometrics, 18, 139–147.

PANDEY, B. N. (1979). Double stage estimation of population variance. Annals of Institute of Statistics Mathematical, 31, 225–233.

PANDEY, B. N. (1983). Shrinkage estimation of the Exponential scale parameter. IEEE Transaction on Reliability, R–32, 203–205.

PANDEY, B. N. (1997). Testimator of the scale parameter of the Exponential distribution using LINEX loss function. Communication in Statistics – Theory and Methods, 26, 2191–2200.

PANDEY, B. N and SRIVASTAVA, R. (1987). A shrinkage testimator for scale parameter of an exponential distribution. Microelectron Reliability, 27 (16), 949–951.

SHAH, S. M. (1964). Use a prior knowledge in the estimation of a parameter from double samples. Journal of Indian Statistical Association, 2, 41–51.

STEIN, C. (1945). A two-stage sample test for a linear hypothesis whose power is independent of the variance. Annals of Mathematical Statistics, 16, 243–258.

THOMPSON, J. R. (1968) Some Shrunken techniques for estimating the Mean. Journal of the American Statistical Association, 63, 113–122.

WAIKAR, V. B., SCHUURMANN, F. J. and RAGHUNATHAN, T. E. (1984). On a two stage shrunken testimator of the mean of a normal distribution. Communications in Statistics – Theory and Methods, 13, 1901–1913.

WEISS, L. (1945). On confidence intervals of given length for the mean of a Normal distribution with unknown variance. Annals of Mathematical Statistics, 16, 348–352.